An Extension of Griffiths' Example 3.8

October 17, 2024

In the limit of large distances we have,

$$V \to -E_0 z + C.$$

Griffiths chooses $C = 0$ for the sphere with no net charge. The reason given for this choice comes from symmetry. However, this choice can be made even if the net charge is not zero as the potential is defined only up to an additive constant ($\vec{\mathbf{E}} = -\vec{\nabla} V$). Now, a general case of a sphere with some net charge (say q) can be worked out with only minor changes in the solution presented in the book. The primary change is in the boundary condition equations 3.74.

$$\begin{aligned} V &= V_0, &\quad \text{for } r = R, \\ V &\to -E_0 r \cos\theta &\quad \text{for } r \gg R. \end{aligned}$$

The first conditions uses the fact that a conductor is an equipotential under static conditions. V_0 is the potential of this equipotential. Hence the first condition gives,

$$\begin{aligned} A_l R^l + \frac{B_l}{R^{l+1}} &= 0, &\qquad \text{for } l > 0 \\ A_0 + \frac{B_0}{R} &= V_0, &\qquad \text{for } l = 0, \end{aligned}$$

as $P_0(\cos\theta) = 1$. Hence,

$$\begin{aligned} B_l &= -A_l R^{2l+1}, &\qquad \text{for } l > 0 \\ B_0 &= (V_0 - A_0) R, &\qquad \text{for } l = 0, \end{aligned}$$

This is the changed version of equation 3.75 of the book. Hence,

$$V(r,\theta) = A_0 + \frac{(V_0 - A_0)R}{r} + \sum_{l=1}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta).$$

For $r \gg R$, all terms with r in the denominator must vanish. This leaves,

$$V(r,\theta) = A_0 + \sum_{l=1}^{\infty} A_l r^l P_l(\cos\theta).$$

However, the boundary condition at $r \gg R$ requires that only $P_1(\cos\theta) = \cos\theta$ remain in the summation. Hence,

$$\begin{aligned} A_0 &= 0 \\ A_1 &= -E_0, \end{aligned}$$

and all other A_l's must be zero. Hence,

$$V(r,\theta) = \frac{V_0 R}{r} - E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta.$$

Then the surface charge density is,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 \left(\frac{V_0}{R} + 3E_0 \cos\theta \right)$$

Hence, the net charge on the surface must be,

$$q = \oint \sigma \; da = \int_0^{2\pi} \int_0^{\pi} \sigma R^2 \sin\theta \; d\theta \; d\phi = 2\pi\epsilon_0 \int_0^{\pi} \left(\frac{V_0}{R} + 3E_0 \cos\theta \right) R^2 \sin\theta \; d\theta = 4\pi\epsilon_0 V_0 R.$$

This relates the surface potential to the charge q as follows.

$$V_0 = \frac{q}{4\pi\epsilon_0 R}.$$