A Visual Proof of Gauss' Theorem

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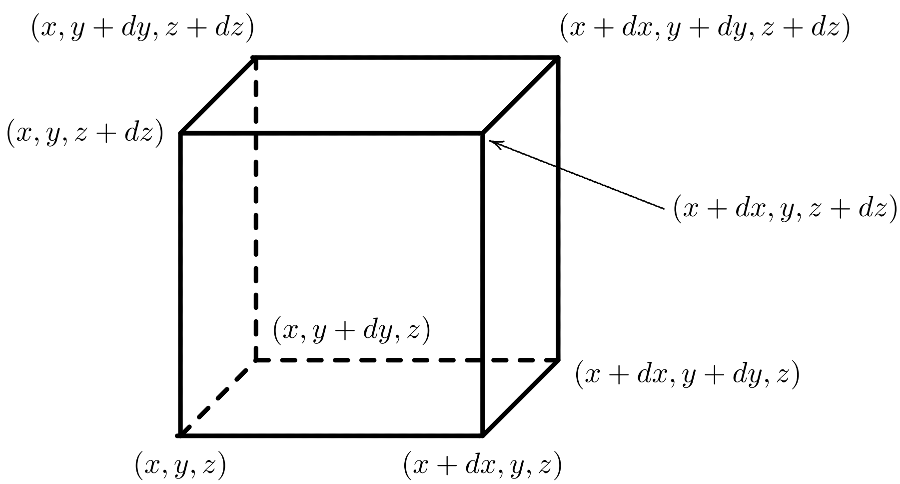
Gauss' theorem states the following for a vector field $\vec{\mathbf{v}}$ in a volume τ contained within a closed surface $\mathcal{S}$.

$$\oint_{\mathcal{S}} \vec{\mathbf{v}} \cdot d\vec{\mathbf{a}} = \int_{\tau} \vec{\nabla} \cdot \vec{\mathbf{v}} d\tau, \tag{1}$$

where $d\vec{\mathbf{a}}$ is an area element of the surface $\mathcal{S}$.

To start the proof, let us consider the following infinitesimal box of sides dx, dy and dz along the three coordinate directions x, y and z and prove the theorem for this box.



The vector $d\vec{\mathbf{a}}$ on the left face of the cube points left (outwards from volume). So, its direction is in the $-\hat{x}$ direction. The magnitude of the area is $dy\, dz$ (as seen by the coordinates shown in the figure). So, the surface integral on the left face is,

$$\vec{\mathbf{v}} \cdot d\vec{\mathbf{a}}_l = (v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) \cdot (-dy\, dz\hat{x}) = -v_x dy\, dz = -v_x(x, y, z) dy\, dz. \tag{2}$$

The functional dependence of v_x has been written explicitly in the last step for a reason that will be seen in the following. The vector $d\vec{\mathbf{a}}$ on the right face of the cube points right (outwards form volume). So, its direction is in the $+\hat{x}$ direction. The computation of flux on this face is similar to the last one but with two distinct differences. There is, of course, no negative sign. Also, v_x being on the right face, it needs to be evaluated at $x+dx$. Hence it becomes,

$$\vec{\mathbf{v}} \cdot d\vec{\mathbf{a}}_r = (v_x\hat{x} + v_y y + \hat{} v_z\hat{z}) \cdot (dy\, dz\hat{x}) = v_x dy\, dz = v_x(x+dx, y, z)dy\, dz. \tag{3}$$

On adding the contributions from the left and the right faces we get,

$$\vec{\mathbf{v}} \cdot d\vec{\mathbf{a}}_{rl} = (v_x(x+dx, y, z) - v_x(x, y, z))dy\, dz = \frac{\partial v_x}{\partial x} dx\, dy\, dz \tag{4}$$

Here the following definition of the derivative is used.

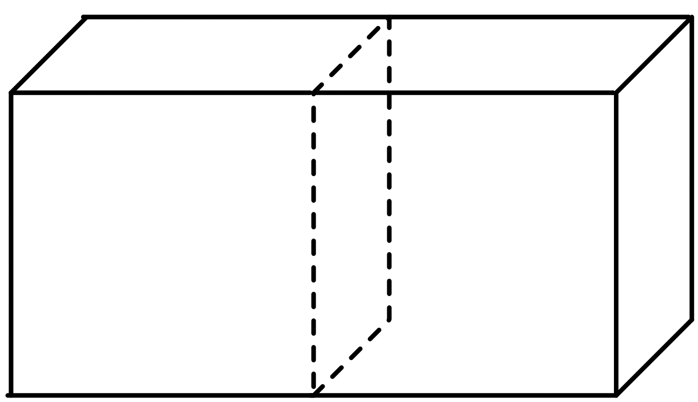
$$\frac{\partial v_x}{\partial x} = \frac{v_x(x+dx, y, z) - v_x(x, y, z)}{dx}. \tag{5}$$

Computation of flux for the other two pairs of faces produce similar results in the y and z directions. So, the total flux through the closed surface of the box is,

$$\vec{\mathbf{v}} \cdot d\vec{\mathbf{a}} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\, dy\, dz = \vec{\nabla} \cdot \vec{\mathbf{v}} d\tau. \tag{6}$$

This proves the theorem for an infinitesimal box.

Now we draw another identical box on the right side of the existing box such that its left face is the same as the right face of the first box. In the following, the shared face is shown by a dashed outline.



For this second box, we can do a similar computation as the first one. Its left face computation is the same as the right face computation for the first box except for the direction of $d\vec{\mathbf{a}}$ – it is the opposite as outwards for the left face is to the left. Hence, between the two boxes, the shared face contributes a zero for the surface integral. So, adding the surface integrals of the two boxes produces the surface integral over the outer surface of the combination of the two boxes. If we build the whole volume τ by stacking such boxes, contributions to the surface integral from shared faces will all cancel leaving only the contributions from the outer faces that are not shared. Hence, the surface integral for any finite region will be only from the outer exposed surface. The volume integral over the same region will simply be the sum of the volume integrals over all infinitesimal boxes. Hence, we get the final result of,

$$\oint_{\mathcal{S}} \vec{\mathbf{v}} \cdot d\vec{\mathbf{a}} = \int_{\tau} \vec{\nabla} \cdot \vec{\mathbf{v}} d\tau, \tag{7}$$