Solutions
Chapter 7

Problem 7

Part a

Considering the counterclockwise direction as positive for emf gives the direction of $d\vec{\mathbf{a}}$ to be out of the page. So,

$$\Phi = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = \int B\, da \cos(180^\circ) = -B \int da = -Blx,$$

where x is the distance of the bar from the left edge so that lx is the area of the loop. This gives the emf to be

$$\mathcal{E} = -\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv.$$

Hence, the current is

$$I = \frac{\mathcal{E}}{R} = \frac{Blv}{R}.$$

This is positive and hence must be counterclockwise (downwards in the resistor).

Part b

The magnetic force is

$$\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}.$$

The current in the bar is upwards and hence,using the right-hand-rule, the direction of $\vec{\mathbf{F}}$ is to the left. The magnitude of $\vec{\mathbf{F}}$ is

$$F = IlB\sin(90^\circ) = \frac{B^2l^2v}{R}.$$

Part c

Applying Newton's second law to the above equation gives

$$m\frac{dv}{dt} = -\frac{B^2l^2v}{R}$$

This first order differential equation has the solution

$$v = v_0 e^{-\alpha t},$$

where

$$\alpha = \frac{B^2l^2}{mR}.$$

Part d

Using the result of the last part, the power loss in the resistor is

$$P = I^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{B^2l^2v^2}{R} = \frac{B^2l^2}{R}v_0^2 e^{-2\alpha t}.$$

The total energy delivered to the resistor is the integral of P up to $t = \infty$.

$$W = \int_0^\infty P\,dt = \int_0^\infty \frac{B^2l^2v_0^2}{R}e^{-2\alpha t}dt = \frac{B^2l^2v_0^2}{2R\alpha}\left(-e^{-2\alpha t}\right)\Big|_0^\infty = \frac{B^2l^2v_0^2}{2R\alpha} = \frac{B^2l^2v_0^2 mR}{2RB^2l^2} = \frac{1}{2}mv_0^2.$$

Problem 16

Using the result of problem 5.11, the magnetic field is found to be in the z direction with a magnitude of

$$B = \mu_0 n I.$$

From symmetry, it is seen that the electric field must be along the $\hat{\phi}$ direction. Choosing the loop integral for emf to be in the $\hat{\phi}$ direction and a circular loop of radius s,

$$\Phi = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = \int B\,|d\vec{\mathbf{a}}|.$$

For points inside the solenoid ($s < a$),

$$\Phi = B\int |d\vec{\mathbf{a}}| = B(\pi s^2) = \mu_0 n I(\pi s^2) = \mu_0 \pi n s^2 I.$$

This gives (Faraday's law)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi}{dt} = -\mu_0 \pi n s^2 \frac{dI}{dt}.$$

Using the cylindrical symmetry,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \oint E\,dl = E\oint dl = E(2\pi s).$$

So,

$$E(2\pi s) = -\mu_0 \pi n s^2 \frac{dI}{dt}.$$

And,

$$E = -\frac{\mu_0 n s}{2}\frac{dI}{dt}.$$

For points outside the solenoid ($s > a$), the magnetic field exists only inside the solenoid and hence the flux integral has to be only within the solenoid. This gives

$$\Phi = B\int |d\vec{\mathbf{a}}| = B(\pi a^2) = \mu_0 n I(\pi a^2) = \mu_0 \pi n a^2 I.$$

So, from Faraday's law,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi}{dt} = -\mu_0 \pi n a^2 \frac{dI}{dt}.$$

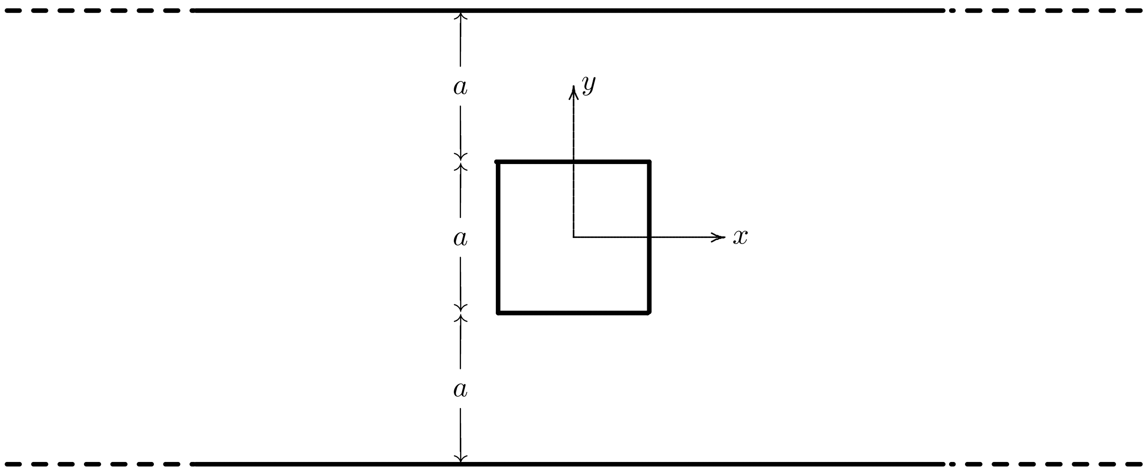
Then

$$E(2\pi s) = -\mu_0 \pi n a^2 \frac{dI}{dt}.$$

And,

$$E = -\frac{\mu_0 n a^2}{2s}\frac{dI}{dt}.$$

Problem 24



Let the small loop be numbered 1 and the big loop numbered 2. So,

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}.$$

The computation of M_{21} requires the computation of the magnetic field and flux due to the small loop. This is not straightforward. However, as $M_{21} = M_{12}$, we can compute M_{12} instead.

$$M_{12} = \Phi_{12}/I_2,$$

where Φ_{12} is the flux in loop 1 due to the current in loop 2.

$$\Phi_{12} = \int \vec{\mathbf{B}}_2 \cdot d\vec{\mathbf{a}}_1.$$

$\vec{\mathbf{B}}_2$ is the magnetic field due to the two long wires together. Using the center of the square as origin and the coordinates as shown above,

$$B_2 = \frac{\mu_0 I_2}{2\pi(3a/2 - y)} + \frac{\mu_0 I_2}{2\pi(3a/2 + y)}$$

So,

$$\Phi_{12} = \int \vec{\mathbf{B}}_2 \cdot d\vec{\mathbf{a}}_1 = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} B_2 dx\, dy = a\int_{-a/2}^{a/2}\left(\frac{\mu_0 I_2}{2\pi(3a/2-y)} + \frac{\mu_0 I_2}{2\pi(3a/2+y)}\right) dy.$$

Integrating, this gives

$$\Phi_{12} = \frac{\mu_0 I_2 a \ln 2}{\pi}.$$

Hence,

$$M_{21} = M_{12} = \Phi_{12}/I_2 = \frac{\mu_0 a \ln 2}{\pi}.$$

And, the induced emf in the big loop is

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt} = -\frac{\mu_0 a \ln 2}{\pi}\frac{dI_1}{dt} = -\frac{\mu_0 a k \ln 2}{\pi}.$$

The negative sign means the emf is counterclockwise (opposite to the direction of the current in the small loop).

Problem 25

The magnetic field in the solenoid is $B = \mu_0 n I$. So the flux through one loop is

$$\Phi_1 = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = \int B|d\vec{\mathbf{a}}| = B\int |d\vec{\mathbf{a}}| = B(\pi R^2) = \mu_0 \pi R^2 n I.$$

So, the flux through N loops is

$$\Phi = N\Phi_1 = N\mu_0\pi R^2 n I.$$

The self inductance of N loops is

$$L = \frac{\Phi}{I} = N\mu_0\pi R^2 n.$$

If the length of N loops is l, then the self inductance per unit length is

$$\frac{L}{l} = \frac{N}{l}\mu_0\pi R^2 n = \mu_0\pi R^2 n^2,$$

as $n = N/l$.